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Automorphisms of real four-dimensional Lie algebras and the invariant characterization of homogeneous 4-spaces

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Abstract
The automorphisms of all four-dimensional, real Lie algebras are presented in a comprehensive way. Their action on the space of 4×4, real, symmetric and positive definite matrices defines equivalence classes which are used for the invariant characterization of the four-dimensional homogeneous spaces which possess an invariant basis.

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1. Introduction

The automorphisms of three-dimensional, real Lie algebras [1] have been proved a powerful tool for analysing the dynamics of 3 + 1 Bianchi cosmological models [2]. At the classical level, time-dependent automorphism inducing diffeomorphisms can be used to simplify the line element—and thus Einstein's field equations—without loss of generality [3]. They also provide an algorithm for counting the number of essential constants; the results obtained agree for all Bianchi types with the preexisting results [4] but, unlike these, the algorithm can be extended to four or more dimensions. At the quantum level, outer automorphisms provide the integrals of motion of the classical Hamiltonian dynamics; their quantum analogues can be used to reconcile quantum Hamiltonian dynamics with the kinematics of homogeneous 3-spaces [5].

A corresponding analysis of these issues for the case of 4 + 1 spatially homogeneous geometries seems very interesting in itself. It could also prove valuable for the nowadays fashionable braneworld models. As a first step in implementing such an analysis, we exhibit the automorphisms for all real, four-dimensional Lie algebras (a first treatise on the subject can be found in [6]) and subsequently use them to invariantly describe homogeneous 4-spaces.

2. Automorphisms

Before exhibiting the results on the automorphisms and their generators, we briefly recall some basic elements of the theory of Lie groups. Topological issues will not concern us, since at this stage of study, they are rather irrelevant.

Let V_N, be a vector space over the field $\mathbb{R}$. For each point x^m in the space, a set of transformations $\widetilde{x}^m = f^m(x^n; \alpha^\mu)$ depending on some parameters α^μ (with Greek indices ranging on the closed interval $[1, \ldots, M]$, while Latin ones, on the closed interval $[1, \ldots, N]$) is defined, endowed with the following properties:

- The parameters are essential, i.e. they are not functions of others; rather, they take values on a compact domain.
- There are particular values for each and every parameter α^μ—which without loss of generality can be taken to be all zero—such that $x^m = f^m(x^n; 0, \ldots, 0)$. In other words, the identity transformation is reached continuously when all the parameters reach this particular set of values (here the zeros).
- The Jacobian of the transformation, $J^m_n = |\partial f^m(x^k; \alpha^\mu)/\partial x^n|$, is non-vanishing on its entire domain of definition. Therefore, every transformation, at least locally, must be invertible.

If one assumes that the parameters are small, and expands in Taylor series the transformations (the functions f^m are taken to be C^n differentiable, with n depending on the application) one obtains

$$\widetilde{x}^m = f^m(x^k; 0, \ldots, 0) + \alpha^\mu \left(\frac{\partial f^m(x^k; \alpha^\nu)}{\partial \alpha^\mu}\bigg|_{\alpha^\nu=0} \right) + \mathcal{O}(\alpha^\mu \alpha^\nu) \tag{2.1}$$

(the Einstein summation convention is in use). Then, a set of M N-dimensional vector fields (each for every essential parameter) is associated with the previous infinitesimal transformations:

$$X^m_\mu = \frac{\partial f^m(x^n; \alpha^\nu)}{\partial \alpha^\mu}\bigg|_{\alpha^\nu=0}. \tag{2.2}$$

These vector fields are called 'generators' and form an algebra:

$$[X_\mu, X_\nu] = C^\kappa_{\mu\nu} X_\kappa \tag{2.3}$$

which is called a Lie algebra—due to the above-mentioned properties. If the quantities $C^\kappa_{\mu\nu}$ do not depend on the space point x^m, they are called 'structure constants', otherwise 'structure functions' and the corresponding algebras, open Lie algebras. In what follows, the Lie algebras are assumed to be closed. In this case, the vector space V_N admits a group of motions (i.e. transformations) G_M with which the Lie algebra of the generators of the transformations is associated. Then it can be proved that $M < N(N+1)/2$ (see [7] for a detailed analysis).

The Jacobi identities for the generators hold:

$$[[X_\mu, X_\nu], X_\kappa] + [[X_\nu, X_\kappa], X_\mu] + [[X_\kappa, X_\mu], X_\nu] = 0 \tag{2.4}$$

or in terms of the structure constants

$$C^\rho_{\mu\nu} C^\sigma_{\rho\kappa} + C^\rho_{\nu\kappa} C^\sigma_{\rho\mu} + C^\rho_{\kappa\mu} C^\sigma_{\rho\nu} = 0. \tag{2.5}$$

If one contracts the index σ with a covariant index, e.g. κ, one gets the contracted Jacobi identities:

$$C^\rho_{\mu\nu} C^\sigma_{\rho\sigma} = 0 \tag{2.6}$$

and thus a 'natural' quantity emerges, namely $C^\sigma_{\rho\sigma} \doteq \nu_\rho$, which is a covector under the action of $GL(M, \mathbb{R})$ (see [8] and references therein).

Under a linear mixing, i.e. the action of the $GL(M, \mathbb{R})$, of the generators:

$$X_\nu \to \widetilde{X}_\nu = L^\mu_\nu X_\mu \tag{2.7}$$

the structure constants transform, according to (2.3), as

$$C^\kappa_{\mu\nu} \to \widetilde{C}^\kappa_{\mu\nu} = L^\alpha_\mu L^\beta_\nu (L^{-1})^\kappa_\rho C^\rho_{\alpha\beta} \tag{2.8}$$

while

$$v_\nu \to \widetilde{v}_\nu = L^\mu_\nu v_\mu. \tag{2.9}$$

The subset of these transformations (i.e. of the form (2.7)) with respect to which the structure constants are invariant is the automorphism group of the Lie algebra $Aut(G)$. If Λ^μ_ν are the matrices of this group, then

$$C^\kappa_{\mu\nu} = \Lambda^\alpha_\mu \Lambda^\beta_\nu (\Lambda^{-1})^\kappa_\rho C^\rho_{\alpha\beta}. \tag{2.10}$$

At first sight, in order to find the automorphism group of a given Lie algebra (i.e. for a given set of non-vanishing structure constants), one has to solve the cubic system (2.10), which can be transformed to quadratic by noting that the matrices of interest are non-singular and thus

$$C^\kappa_{\mu\nu} \Lambda^\rho_\kappa = \Lambda^\alpha_\mu \Lambda^\beta_\nu C^\rho_{\alpha\beta} \tag{2.11}$$

but still, the quest for the solutions remains a difficult task.

A first simplification may be achieved by observing that

$$v_\nu = \Lambda^\mu_\nu v_\mu. \tag{2.12}$$

A second simplification makes use of the Killing–Cartan metric—provided by a famous theorem due to Cartan:

$$g_{\mu\nu} = C^\alpha_{\beta\mu} C^\beta_{\alpha\nu}. \tag{2.13}$$

Automorphisms preserve its form, i.e. are isometries of this metric:

$$g_{\mu\nu} = \Lambda^\alpha_\mu \Lambda^\beta_\nu g_{\alpha\beta}. \tag{2.14}$$

The combined use of (2.11), (2.12) and (2.14) makes the first simpler to solve, providing us with useful necessary conditions restricting the Λ^μ_ν.

In order to find the generators of the automorphism group of a given Lie algebra, it is necessary to consider a family of automorphic matrices which is connected to the identity, i.e. $\Lambda^\mu_\nu = \Lambda^\mu_\nu(\tau)$ for some parameter τ such that $\Lambda^\mu_\nu(0) = I_M$, with I_M the M-dimensional identity matrix. Then if one substitutes this family into (2.11), differentiates with respect to this parameter, and sets to zero, one will get

$$\lambda^\kappa_\rho C^\rho_{\mu\nu} = \lambda^\rho_\mu C^\kappa_{\rho\nu} + \lambda^\rho_\nu C^\kappa_{\mu\rho} \tag{2.15}$$

where

$$\lambda^\mu_\nu = \left.\frac{\mathrm{d}\Lambda^\mu_\nu(\tau)}{\mathrm{d}\tau}\right|_{\tau=0} \tag{2.16}$$

is the required generator. The system (2.15) is linear and thus easy to solve when the values of the structure constants are given. The number of independent solutions to it determines the number of independent parameters of the generators of the automorphism group.

The situation in the literature concerning the four-dimensional, real Lie algebras is characterized by a certain degree of diversity. The main reason is that, unlike the case of three-dimensional, real Lie algebras, a unique decomposition of the structure constants' tensor in terms of lower rank objects has not been found. As a result, the presentations of

Petrov [7], MacCallum [8] and Patera *et al* [9] differ substantially, especially as far as the number of different real Lie algebras is concerned.

In the following, the non-vanishing structure constants for the various four-dimensional, real Lie algebras (according to Patera *et al* [9] which is considered to be the most complete and extensive exposition), the automorphism matrices and their generators are given in table 1. Also, for each algebra, an irreducible form of a generic, 4×4, symmetric, positive definite real matrix is given in table 2 along with a suggested basis of invariants.

We now come to the invariant description of a homogeneous 4-space. Let $\sigma_i^\alpha(x)$ denote the basis of 1-forms, invariant under the action of the symmetry group of motions, acting simply transitively on the space. Then

$$\sigma^\alpha_{i,j}(x) - \sigma^\alpha_{j,i}(x) = 2C^\alpha_{\mu\nu}\sigma^\mu_j(x)\sigma^\nu_i(x) \tag{2.17}$$

where $C^\alpha_{\mu\nu}$ are the structure constants of the corresponding Lie algebra. Using this basis we can write in these coordinates the most general, manifestly invariant, line element as

$$\mathrm{d}s^2 = \gamma_{\alpha\beta}\sigma^\alpha_i(x)\sigma^\beta_j(x)\,\mathrm{d}x^i\,\mathrm{d}x^j \tag{2.18}$$

where $\gamma_{\alpha\beta}$ is a numerical, 4×4, real, positive definite symmetric matrix. If we consider the class of GCTs $x^i = f^i(y^m)$ which leave the given basis 1-forms quasi-form invariant, i.e. those satisfying

$$\sigma^\alpha_i(x)\frac{\partial x^i}{\partial y^m} = \Lambda^\alpha_\mu\sigma^\mu_m(y) \tag{2.19}$$

then we have a well-defined non-trivial action on the configuration space spanned by $\gamma_{\alpha\beta}$, given by

$$\widetilde{\gamma}_{\mu\nu} = \Lambda^\alpha_\mu\Lambda^\beta_\nu\gamma_{\alpha\beta}. \tag{2.20}$$

The relevant results for 3-spaces are given in [5] and the generalization to 4-spaces is obvious. The requirement for Λ^α_β to be constant leads, through the integrability conditions for (2.19), to the restrictions

$$C^\rho_{\mu\nu}\Lambda^\alpha_\rho = \Lambda^\kappa_\mu\Lambda^\sigma_\nu C^\alpha_{\kappa\sigma} \tag{2.21}$$

which reveal Λ^α_β as an element of the automorphism group of the corresponding Lie algebra. Thus, the configuration space is divided into equivalence classes by the action of the automorphism group according to (2.20).

In order to have an infinitesimal description of this action, we need to consider the generators λ^α_β of Λ^α_β. Their defining relations are (2.15). We can easily see that the linear vector fields in the configuration space:

$$X_{(i)} = \lambda^\alpha_{\mu(i)}\gamma_{\alpha\nu}\frac{\partial}{\partial\gamma_{\mu\nu}} \tag{2.22}$$

induce, through their integral curves, exactly the motions (2.20)—(i) is a collective index corresponding to a choice of base for λ^α_μ and counts the number of independent vector fields. Thus, if we wish for a scalar function $\Psi = \Psi(\gamma_{\alpha\beta})$ to change only when we move from one class to another, then we must demand

$$X_{(i)}\Psi = 0 \qquad \forall i \in [1, \ldots, d] \quad d < 10. \tag{2.23}$$

The solutions to this system of equations, say $q^A = q^A\left(C^\alpha_{\mu\nu}, \gamma_{\mu\nu}\right)$, lead to the finite description of the action of automorphisms—with A taking its values on the interval $[1, \ldots, 10 - d]$. By construction, they satisfy

$$q^{(1)A} = q^{(2)A} \qquad \text{for every} \quad \left(\gamma^{(1)}_{\alpha\beta}, \gamma^{(2)}_{\alpha\beta}\right) \tag{2.24}$$

connected through an automorphism, as in (2.20). A kind of inverse to this proposition, which completes the finite description of the 4-spaces, discussed here, is contained in the following.

Table 1. The first column gives the names of the Lie algebras according to [9]. The second gives the values of the non-vanishing structure constants. Column three exhibits the corresponding automorphism group matrices; in the majority of cases these matrices are connected to the identity matrix. When the automorphism group has a component disconnected from the identity element, the corresponding matrices are not connected to the identity matrix, and are also shown. Finally, in column four, the generators of the automorphism group are presented.

Lie algebra	Non-vanishing structure constants	Automorphisms Λ^{α}_{β}	Generators λ^{α}_{β}
$4A_1$		$GL(4,\mathbb{R})$	$GL(4,\mathbb{R})$
$A_2 \oplus A_1$	$C^2_{12} = 1$	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ a_5 & a_6 & 0 & 0 \\ a_9 & 0 & a_{11} & a_{12} \\ a_{13} & 0 & a_{15} & a_{16} \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ g_5 & g_6 & 0 & 0 \\ g_9 & 0 & g_{11} & g_{12} \\ g_{13} & 0 & g_{15} & g_{16} \end{pmatrix}$
$2A_2$	$C^2_{12} = 1, C^4_{34} = 1$	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ a_5 & a_6 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & a_{15} & a_{16} \end{pmatrix}$ or $\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & a_7 & a_8 \\ 1 & 0 & 0 & 0 \\ a_{13} & a_{14} & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ g_5 & g_6 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & g_{15} & g_{16} \end{pmatrix}$
$A_{3,1} \oplus A_1$	$C^1_{23} = 1$	$\begin{pmatrix} a_{11}a_6 - a_{10}a_7 & a_2 & a_3 & a_4 \\ 0 & a_6 & a_7 & 0 \\ 0 & a_{10} & a_{11} & 0 \\ 0 & a_{14} & a_{15} & a_{16} \end{pmatrix}$	$\begin{pmatrix} g_6 + g_{11} & g_2 & g_3 & g_4 \\ 0 & g_6 & g_7 & 0 \\ 0 & g_{10} & g_{11} & 0 \\ 0 & g_{14} & g_{15} & g_{16} \end{pmatrix}$
$A_{3,2} \oplus A_1$	$C^1_{13} = 1, C^1_{23} = 1, C^2_{23} = 1$	$\begin{pmatrix} a_1 & a_2 & a_3 & 0 \\ 0 & a_1 & a_7 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & a_{15} & a_{16} \end{pmatrix}$	$\begin{pmatrix} g_1 & g_2 & g_3 & 0 \\ 0 & g_1 & g_7 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & g_{15} & g_{16} \end{pmatrix}$
$A_{3,3} \oplus A_1$	$C^1_{13} = 1, C^2_{23} = 1$	$\begin{pmatrix} a_1 & a_2 & a_3 & 0 \\ a_5 & a_6 & a_7 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & a_{15} & a_{16} \end{pmatrix}$	$\begin{pmatrix} g_1 & g_2 & g_3 & 0 \\ g_5 & g_6 & g_7 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & g_{15} & g_{16} \end{pmatrix}$
$A_{3,4} \oplus A_1$	$C^1_{13} = 1, C^2_{23} = -1$	$\begin{pmatrix} a_1 & 0 & a_3 & 0 \\ 0 & a_6 & a_7 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & a_{15} & a_{16} \end{pmatrix}$ or $\begin{pmatrix} 0 & a_2 & a_3 & 0 \\ a_5 & 0 & a_7 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & a_{15} & a_{16} \end{pmatrix}$	$\begin{pmatrix} g_1 & 0 & g_3 & 0 \\ 0 & g_6 & g_7 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & g_{15} & g_{16} \end{pmatrix}$
$A^{\alpha}_{3,5} \oplus A_1, 0 < \lvert\alpha\rvert < 1$	$C^1_{13} = 1, C^2_{23} = \alpha$	$\begin{pmatrix} a_1 & 0 & a_3 & 0 \\ 0 & a_6 & a_7 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & a_{15} & a_{16} \end{pmatrix}$	$\begin{pmatrix} g_1 & 0 & g_3 & 0 \\ 0 & g_6 & g_7 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & g_{15} & g_{16} \end{pmatrix}$

Table 1. (Continued.)

Lie algebra	Non-vanishing structure constants	Automorphisms Λ^{α}_{β}	Generators λ^{α}_{β}
$A_{3,6} \oplus A_1$	$C^2_{13} = -1, C^1_{23} = 1$	$\begin{pmatrix} a_1 & a_2 & a_3 & 0 \\ -a_2 & a_1 & a_7 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & a_{15} & a_{16} \end{pmatrix}, \begin{pmatrix} a_1 & a_2 & a_3 & 0 \\ a_2 & -a_1 & a_7 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & a_{15} & a_{16} \end{pmatrix}$	$\begin{pmatrix} g_1 & g_2 & g_3 & 0 \\ -g_2 & g_1 & g_4 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & g_{15} & g_{16} \end{pmatrix}$
$A^{\alpha}_{3,7} \oplus A_1, 0 < \alpha$	$C^1_{13} = \alpha, C^2_{13} = -1,$ $C^1_{23} = 1, C^2_{23} = \alpha$	$\begin{pmatrix} a_1 & a_2 & a_3 & 0 \\ -a_2 & a_1 & a_7 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & a_{15} & a_{16} \end{pmatrix}$	$\begin{pmatrix} g_1 & g_2 & g_3 & 0 \\ -g_2 & g_1 & g_7 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & g_{15} & g_{16} \end{pmatrix}$
$A_{3,8} \oplus A_1$	$C^1_{23} = 1, C^2_{13} = -1, C^3_{12} = -1$	See the appendix	
$A_{3,9} \oplus A_1$	$C^3_{12} = 1, C^1_{23} = 1, C^2_{31} = 1$	See the appendix	
$A_{4,1}$	$C^1_{24} = 1, C^2_{34} = 1$	$\begin{pmatrix} a_{11}a^2_{16} & a_7 a_{16} & a_3 & a_4 \\ 0 & a_{11}a_{16} & a_7 & a_8 \\ 0 & 0 & a_{11} & a_{12} \\ 0 & 0 & 0 & a_{16} \end{pmatrix}$	$\begin{pmatrix} g_{11} + 2g_{16} & g_7 & g_3 & g_4 \\ 0 & g_{11} + g_{16} & g_7 & g_8 \\ 0 & 0 & g_{11} & g_{12} \\ 0 & 0 & 0 & g_{16} \end{pmatrix}$
$A^{\alpha}_{4,2}, \alpha \neq (0, 1)$	$C^1_{14} = \alpha, C^2_{24} = 1,$ $C^2_{34} = 1, C^3_{34} = 1$	$\begin{pmatrix} a_1 & 0 & 0 & a_4 \\ 0 & a_6 & 0 & a_8 \\ 0 & 0 & a_6 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} g_1 & 0 & 0 & g_4 \\ 0 & g_6 & 0 & g_8 \\ 0 & 0 & g_6 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$
$A^1_{4,2}$	$C^1_{14} = 1, C^2_{24} = 1,$ $C^2_{34} = 1, C^3_{34} = 1$	$\begin{pmatrix} a_1 & 0 & 0 & a_4 \\ a_5 & a_6 & 0 & a_8 \\ 0 & 0 & a_6 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} g_1 & 0 & 0 & g_4 \\ g_5 & g_{11} & 0 & g_8 \\ 0 & 0 & g_{11} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$
$A_{4,3}$	$C^1_{14} = 1, C^2_{34} = 1$	$\begin{pmatrix} a_1 & 0 & 0 & a_4 \\ 0 & a_6 & a_7 & a_8 \\ 0 & 0 & a_6 & a_{12} \\ 0 & 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} g_1 & 0 & 0 & g_4 \\ 0 & g_{11} & g_7 & g_8 \\ 0 & 0 & g_{11} & g_{12} \\ 0 & 0 & 0 & 0 \end{pmatrix}$
$A_{4,4}$	$C^1_{14} = 1, C^1_{24} = 1,$ $C^2_{24} = 1, C^2_{34} = 1, C^3_{34} = 1$	$\begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ 0 & a_1 & a_2 & a_8 \\ 0 & 0 & a_1 & a_{12} \\ 0 & 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} g_1 & g_7 & g_3 & g_4 \\ 0 & g_1 & g_7 & g_8 \\ 0 & 0 & g_1 & g_{12} \\ 0 & 0 & 0 & 0 \end{pmatrix}$

Table 1. (Continued.)

Lie algebra	Non-vanishing structure constants	Automorphisms Λ_β^α	Generators λ_β^α
$A_{4,5}^{\alpha,\beta}, -1 \leqslant \alpha < \beta < 1, \alpha\beta \neq 0$	$C_{14}^1 = 1, C_{24}^2 = \alpha, C_{34}^3 = \beta$	$\begin{pmatrix} a_1 & 0 & 0 & a_4 \\ 0 & a_6 & 0 & a_8 \\ 0 & 0 & a_{11} & a_{12} \\ 0 & 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} g_1 & 0 & 0 & g_4 \\ 0 & g_6 & 0 & g_8 \\ 0 & 0 & g_{11} & g_{12} \\ 0 & 0 & 0 & 0 \end{pmatrix}$
$A_{4,5}^{\alpha,\alpha}, -1 \leqslant \alpha < 1, \alpha \neq 0$	$C_{14}^1 = 1, C_{24}^2 = \alpha, C_{34}^3 = \alpha$	$\begin{pmatrix} a_1 & 0 & 0 & a_4 \\ 0 & a_6 & a_7 & a_8 \\ 0 & a_{10} & a_{11} & a_{12} \\ 0 & 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} g_1 & 0 & 0 & g_4 \\ 0 & g_6 & g_7 & g_8 \\ 0 & g_{10} & g_{11} & g_{12} \\ 0 & 0 & 0 & 0 \end{pmatrix}$
$A_{4,5}^{\alpha,1}, -1 \leqslant \alpha < 1, \alpha \neq 0$	$C_{14}^1 = 1, C_{24}^2 = \alpha, C_{34}^3 = 1$	$\begin{pmatrix} a_1 & 0 & a_3 & a_4 \\ 0 & a_6 & 0 & a_8 \\ a_9 & 0 & a_{11} & a_{12} \\ 0 & 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} g_1 & 0 & g_3 & g_4 \\ 0 & g_6 & 0 & g_8 \\ g_9 & 0 & g_{11} & g_{12} \\ 0 & 0 & 0 & 0 \end{pmatrix}$
$A_{4,5}^{1,1}$	$C_{14}^1 = 1, C_{24}^2 = 1, C_{34}^3 = 1$	$\begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \\ 0 & 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} g_1 & g_2 & g_3 & g_4 \\ g_5 & g_6 & g_7 & g_8 \\ g_9 & g_{10} & g_{11} & g_{12} \\ 0 & 0 & 0 & 0 \end{pmatrix}$
$A_{4,6}^{\alpha,\beta}, \alpha \neq 0, \beta \geqslant 0$	$C_{14}^1 = \alpha, C_{24}^2 = \beta, C_{24}^3 = -1,$ $C_{34}^2 = 1, C_{34}^3 = \beta$	$\begin{pmatrix} a_1 & 0 & 0 & a_4 \\ 0 & a_6 & a_7 & a_8 \\ 0 & -a_7 & a_6 & a_{12} \\ 0 & 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} g_1 & 0 & 0 & g_4 \\ 0 & g_{11} & -g_{10} & g_8 \\ 0 & g_{10} & g_{11} & g_{12} \\ 0 & 0 & 0 & 0 \end{pmatrix}$
$A_{4,7}$	$C_{14}^1 = 2, C_{24}^2 = 1, C_{34}^2 = 1,$ $C_{34}^3 = 1, C_{23}^1 = 1$	$\begin{pmatrix} a_6^2 & -a_{12}a_6 & -a_{12}(a_6 + a_7) + a_6a_8 & a_4 \\ 0 & a_6 & a_7 & a_8 \\ 0 & 0 & a_6 & a_{12} \\ 0 & 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 2g_{11} & -g_{12} & -g_{12} + g_8 & g_4 \\ 0 & g_{11} & g_7 & g_8 \\ 0 & 0 & g_{11} & g_{12} \\ 0 & 0 & 0 & 0 \end{pmatrix}$
$A_{4,8}$	$C_{23}^1 = 1, C_{24}^2 = 1, C_{34}^3 = -1$	$\begin{pmatrix} a_{11}a_6 & a_{12}a_6 & a_{11}a_8 & a_4 \\ 0 & a_6 & 0 & a_8 \\ 0 & 0 & a_{11} & a_{12} \\ 0 & 0 & 0 & 1 \end{pmatrix}$ or $\begin{pmatrix} -a_{10}a_7 & -a_{10}a_8 & -a_{12}a_7 & a_4 \\ 0 & 0 & a_7 & a_8 \\ 0 & a_{10} & 0 & a_{12} \\ 0 & 0 & 0 & -1 \end{pmatrix}$	$\begin{pmatrix} g_{11} + g_6 & g_{12} & g_8 & g_4 \\ 0 & g_6 & 0 & g_8 \\ 0 & 0 & g_{11} & g_{12} \\ 0 & 0 & 0 & 0 \end{pmatrix}$

Table 1. (Continued.)

Lie algebra	Non-vanishing structure constants	Automorphisms Λ^{α}_{β}	Generators λ^{α}_{β}
$A^{\beta}_{4,9}, 0 < \lvert\beta\rvert < 1$	$C^1_{23} = 1, C^1_{14} = 1 + \beta,$ $C^2_{24} = 1, C^3_{34} = \beta$	$\begin{pmatrix} a_{11}a_6 & -a_{12}a_6/\beta & a_8a_{11} & a_4 \\ 0 & a_6 & 0 & a_8 \\ 0 & 0 & a_{11} & a_{12} \\ 0 & 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} g_{11} + g_6 & -g_{12}/\beta & g_8 & g_4 \\ 0 & g_6 & 0 & g_8 \\ 0 & 0 & g_{11} & g_{12} \\ 0 & 0 & 0 & 0 \end{pmatrix}$
$A^1_{4,9}$	$C^1_{23} = 1, C^1_{14} = 2,$ $C^2_{24} = 1, C^3_{34} = 1$	$\begin{pmatrix} a_{11}a_6 - a_{10}a_7 & -a_{12}a_6 + a_{10}a_8 & a_8a_{11} - a_7a_{12} & a_4 \\ 0 & a_6 & a_7 & a_8 \\ 0 & a_{10} & a_{11} & a_{12} \\ 0 & 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} g_{11} + g_6 & -g_{12} & g_8 & g_4 \\ 0 & g_6 & g_7 & g_8 \\ 0 & g_{10} & g_{11} & g_{12} \\ 0 & 0 & 0 & 0 \end{pmatrix}$
$A^0_{4,9}$	$C^1_{23} = 1, C^1_{14} = 1, C^2_{24} = 1$	$\begin{pmatrix} a_{11}a_6 & a_2 & a_8a_{11} & a_4 \\ 0 & a_6 & 0 & a_8 \\ 0 & 0 & a_{11} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} g_{11} + g_6 & g_2 & g_8 & g_4 \\ 0 & g_6 & 0 & g_8 \\ 0 & 0 & g_{11} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$
$A_{4,10}$	$C^1_{23} = 1, C^3_{24} = -1, C^2_{34} = 1$	$\begin{pmatrix} a_6^2 + a_7^2 & a_{12}a_7 - a_6a_8 & -a_{12}a_6 - a_7a_8 & a_4 \\ 0 & a_6 & a_7 & a_8 \\ 0 & -a_7 & a_6 & a_{12} \\ 0 & 0 & 0 & 1 \end{pmatrix}$ or $\begin{pmatrix} -a_6^2 - a_7^2 & a_{12}a_7 + a_6a_8 & -a_{12}a_6 + a_7a_8 & a_4 \\ 0 & a_6 & a_7 & a_8 \\ 0 & a_7 & -a_6 & a_{12} \\ 0 & 0 & 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 2g_{11} & -g_8 & -g_{12} & g_4 \\ 0 & g_{11} & -g_{10} & g_8 \\ 0 & g_{10} & g_{11} & g_{12} \\ 0 & 0 & 0 & 0 \end{pmatrix}$
$A^{\alpha}_{4,11}, \alpha > 0$	$C^1_{23} = 1, C^1_{14} = 2\alpha, C^2_{24} = \alpha,$ $C^3_{24} = -1, C^2_{34} = 1, C^3_{34} = \alpha$	$\begin{pmatrix} a_6^2 + a_7^2 & -(w1)/(1+\alpha^2) & -(w2)/(1+\alpha^2) & a_4 \\ 0 & a_6 & a_7 & a_8 \\ 0 & -a_7 & a_6 & a_{12} \\ 0 & 0 & 0 & 1 \end{pmatrix}$ $w1 = a_6(\alpha\, a_{12} + a_8) + a_7(\alpha\, a_8 - a_{12})$ $w2 = a_6(a_{12} - \alpha\, a_8) + a_7(\alpha\, a_{12} + a_8)$	$\begin{pmatrix} 2g_{11} & g_2 & g_3 & g_4 \\ 0 & g_{11} & -g_{10} & -g_2 + \alpha g_3 \\ 0 & g_{10} & g_{11} & -\alpha g_2 - g_3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$
$A_{4,12}$	$C^1_{13} = 1, C^2_{23} = 1,$ $C^2_{14} = -1, C^1_{24} = 1$	$\begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ -a_2 & a_1 & a_4 & -a_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ or $\begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ a_2 & -a_1 & -a_4 & a_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$	$\begin{pmatrix} g_6 & -g_5 & -g_8 & g_7 \\ g_5 & g_6 & g_7 & g_8 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

Table 2. The first column gives the names of the Lie algebras according to [9]. The second gives a possible irreducible form for a generic 4 × 4, real, symmetric, positive definite matrix resulting from the action of the corresponding automorphism group. Care has been taken, so that the exhibited reduced form can always be achieved. Finally, in column three, a complete set of functionally independent metric invariants is given. Functional independence has been tested using the reduced form of the matrix given in the second column. However, since (a) the reduction has been reached by means of the appropriate automorphism group and (b) the action of this group is nothing but the effect of a general coordinate transformation, one concludes that the given metric invariants are functionally independent even if the components of the generic matrix are considered as their arguments.

Lie algebra	(Possible) reducible form of $\gamma_{\alpha\beta}$	Functionally independent invariants
$4A_1$	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$	None
$A_2 \oplus 2A_1$	$\begin{pmatrix} \gamma_{11} & 0 & 0 & 0 \\ 0 & 1 & 0 & \gamma_{24} \\ 0 & 0 & 1 & 0 \\ 0 & \gamma_{24} & 0 & 1 \end{pmatrix}$	q^1, q^2
$2A_2$	$\begin{pmatrix} \gamma_{11} & 0 & \gamma_{13} & \gamma_{14} \\ 0 & 1 & \gamma_{23} & \gamma_{24} \\ \gamma_{13} & \gamma_{23} & \gamma_{33} & 0 \\ \gamma_{14} & \gamma_{24} & 0 & 1 \end{pmatrix}$	$q^1, q^2, q^3, q^4, q^5, q^6$
$A_{3,1} \oplus A_1$	$\begin{pmatrix} \gamma_{11} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$	q^1
$A_{3,2} \oplus A_1$	$\begin{pmatrix} 1 & 0 & 0 & \gamma_{14} \\ 0 & \gamma_{22} & 0 & \gamma_{24} \\ 0 & 0 & \gamma_{33} & 0 \\ \gamma_{14} & \gamma_{24} & 0 & 1 \end{pmatrix}$	q^1, q^2, q^3, q^5
$A_{3,3} \oplus A_1$	$\begin{pmatrix} 1 & 0 & 0 & \gamma_{14} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \gamma_{33} & 0 \\ \gamma_{14} & 0 & 0 & 1 \end{pmatrix}$	q^1, q^2
$A_{3,4} \oplus A_1$	$\begin{pmatrix} 1 & \gamma_{12} & 0 & \gamma_{14} \\ \gamma_{12} & 1 & 0 & \gamma_{24} \\ 0 & 0 & \gamma_{33} & 0 \\ \gamma_{14} & \gamma_{24} & 0 & 1 \end{pmatrix}$	q^1, q^2, q^3, q^5
$A_{3,5}^{\alpha} \oplus A_1$	$\begin{pmatrix} 1 & \gamma_{12} & 0 & \gamma_{14} \\ \gamma_{12} & 1 & 0 & \gamma_{24} \\ 0 & 0 & \gamma_{33} & 0 \\ \gamma_{14} & \gamma_{24} & 0 & 1 \end{pmatrix}$	q^1, q^2, q^3, q^5
$A_{3,6} \oplus A_1$	$\begin{pmatrix} 1 & 0 & 0 & \gamma_{14} \\ 0 & \gamma_{22} & 0 & \gamma_{24} \\ 0 & 0 & \gamma_{33} & 0 \\ \gamma_{14} & \gamma_{24} & 0 & 1 \end{pmatrix}$	q^1, q^2, q^3, q^5
$A_{3,7}^{\alpha} \oplus A_1$	$\begin{pmatrix} 1 & 0 & 0 & \gamma_{14} \\ 0 & \gamma_{22} & 0 & \gamma_{24} \\ 0 & 0 & \gamma_{33} & 0 \\ \gamma_{14} & \gamma_{24} & 0 & 1 \end{pmatrix}$	q^1, q^3, q^5, q^6

Table 2. (Continued.)

Lie algebra	(Possible) reducible form of $\gamma_{\alpha\beta}$	Functionally independent invariants
$A_{3,8}\oplus A_1$ and $A_{3,9}\oplus A_1$	$\begin{pmatrix}\gamma_{11} & 0 & 0 & \gamma_{14}\\ 0 & \gamma_{22} & 0 & \gamma_{24}\\ 0 & 0 & \gamma_{33} & \gamma_{34}\\ \gamma_{14} & \gamma_{24} & \gamma_{34} & 1\end{pmatrix}$	$q^1, q^2, q^3, q^4, q^5, q^6$
$A_{4,1}$	$\begin{pmatrix}\gamma_{11} & \gamma_{12} & 0 & 0\\ \gamma_{12} & \gamma_{22} & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}$	q^1, q^3, q^5
$A_{4,2}^{\alpha}$	$\begin{pmatrix}1 & \gamma_{12} & \gamma_{13} & 0\\ \gamma_{12} & 1 & \gamma_{23} & 0\\ \gamma_{13} & \gamma_{23} & \gamma_{33} & \gamma_{34}\\ 0 & 0 & \gamma_{34} & \gamma_{44}\end{pmatrix}$	$q^1, q^2, q^3, q^4, q^5, q^6$
$A_{4,2}^{1}$	$\begin{pmatrix}1 & 0 & \gamma_{13} & 0\\ 0 & 1 & \gamma_{23} & 0\\ \gamma_{13} & \gamma_{23} & \gamma_{33} & \gamma_{34}\\ 0 & 0 & \gamma_{34} & \gamma_{44}\end{pmatrix}$	q^1, q^2, q^3, q^4, q^5
$A_{4,3}$	$\begin{pmatrix}1 & \gamma_{12} & \gamma_{13} & 0\\ \gamma_{12} & 1 & 0 & 0\\ \gamma_{13} & 0 & \gamma_{33} & 0\\ 0 & 0 & 0 & \gamma_{44}\end{pmatrix}$	q^1, q^2, q^3, q^5
$A_{4,4}$	$\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & \gamma_{22} & \gamma_{23} & 0\\ 0 & \gamma_{23} & \gamma_{33} & 0\\ 0 & 0 & 0 & \gamma_{44}\end{pmatrix}$	q^1, q^2, q^3, q^5
$A_{4,5}^{\alpha,\beta}$	$\begin{pmatrix}1 & \gamma_{12} & \gamma_{13} & 0\\ \gamma_{12} & 1 & \gamma_{23} & 0\\ \gamma_{13} & \gamma_{23} & 1 & 0\\ 0 & 0 & 0 & \gamma_{44}\end{pmatrix}$	q^1, q^2, q^3, q^5
$A_{4,5}^{\alpha,\alpha}$	$\begin{pmatrix}1 & \gamma_{12} & 0 & 0\\ \gamma_{12} & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & \gamma_{44}\end{pmatrix}$	q^1, q^2
$A_{4,5}^{\alpha,1}$	$\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & \gamma_{23} & 0\\ 0 & \gamma_{23} & 1 & 0\\ 0 & 0 & 0 & \gamma_{44}\end{pmatrix}$	q^1, q^2
$A_{4,5}^{1,1}$	$\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & \gamma_{44}\end{pmatrix}$	q^1
$A_{4,6}^{\alpha,\beta}$	$\begin{pmatrix}1 & \gamma_{12} & \gamma_{13} & 0\\ \gamma_{12} & 1 & 0 & 0\\ \gamma_{13} & 0 & \gamma_{33} & 0\\ 0 & 0 & 0 & \gamma_{44}\end{pmatrix}$	q^1, q^3, q^4, q^5
$A_{4,7}$	$\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & \gamma_{22} & 0 & \gamma_{24}\\ 0 & 0 & \gamma_{33} & \gamma_{34}\\ 0 & \gamma_{24} & \gamma_{34} & \gamma_{44}\end{pmatrix}$	q^1, q^2, q^3, q^4, q^5

Table 2. (Continued.)

Lie algebra	(Possible) reducible form of $\gamma_{\alpha\beta}$	Functionally independent invariants
$A_{4,8}$	$\begin{pmatrix} \gamma_{11} & 0 & 0 & 0 \\ 0 & 1 & \gamma_{23} & \gamma_{24} \\ 0 & \gamma_{23} & 1 & \gamma_{34} \\ 0 & \gamma_{24} & \gamma_{34} & \gamma_{44} \end{pmatrix}$	q^1, q^2, q^3, q^4, q^5
$A_{4,9}^{\beta}$	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \gamma_{23} & \gamma_{24} \\ 0 & \gamma_{23} & \gamma_{33} & \gamma_{34} \\ 0 & \gamma_{24} & \gamma_{34} & \gamma_{44} \end{pmatrix}$	q^1, q^2, q^3, q^4, q^5
$A_{4,9}^{1}$	$\begin{pmatrix} \gamma_{11} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \gamma_{34} \\ 0 & 0 & \gamma_{34} & \gamma_{44} \end{pmatrix}$	q^1, q^2, q^3
$A_{4,9}^{0}$	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \gamma_{23} & \gamma_{24} \\ 0 & \gamma_{23} & \gamma_{33} & \gamma_{34} \\ 0 & \gamma_{24} & \gamma_{34} & \gamma_{44} \end{pmatrix}$	q^1, q^2, q^3, q^4, q^5
$A_{4,10}$	$\begin{pmatrix} \gamma_{11} & 0 & 0 & 0 \\ 0 & 1 & 0 & \gamma_{24} \\ 0 & 0 & \gamma_{33} & \gamma_{34} \\ 0 & \gamma_{24} & \gamma_{34} & \gamma_{44} \end{pmatrix}$	q^1, q^2, q^3, q^4, q^5
$A_{4,11}^{\alpha}$	$\begin{pmatrix} \gamma_{11} & 0 & 0 & 0 \\ 0 & 1 & 0 & \gamma_{24} \\ 0 & 0 & \gamma_{33} & \gamma_{34} \\ 0 & \gamma_{24} & \gamma_{34} & \gamma_{44} \end{pmatrix}$	q^1, q^3, q^4, q^5, q^6
$A_{4,12}$	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \gamma_{22} & \gamma_{23} & \gamma_{33} \\ 0 & \gamma_{23} & \gamma_{33} & \gamma_{34} \\ 0 & \gamma_{24} & \gamma_{34} & \gamma_{44} \end{pmatrix}$	$q^1, q^2, q^3, q^4, q^5, q^6$

Theorem 1. *Let $\gamma_{\alpha\beta}^{(1)}, \gamma_{\alpha\beta}^{(2)}$ belong to the configuration space. If $q^{(1)A} = q^{(2)A}\ \forall A$, then there is $\Lambda_{\beta}^{\alpha} \in Aut(G)$ such that $\gamma_{\mu\nu}^{(2)} = \Lambda_{\mu}^{\alpha}\Lambda_{\nu}^{\beta}\gamma_{\alpha\beta}^{(1)}$.*

Proof. We first observe that, as seen in table 2, for each and every Lie algebra the (connected to the identity) component of the automorphism group suffices to bring the generic, positive definite, real 4×4 $\gamma_{\alpha\beta}$ to an irreducible (though not unique) form $\gamma_{\alpha\beta}^{\text{Ir.}}$, possessing a number of remaining arbitrary components which equals the number of independent $q^{(A)}$s. Thus, in this 'gauge' the hypothesis of the theorem, i.e. $q^{(1)A} = q^{(2)A}\ \forall A$, implies through the implicit function theorem [10] that $\gamma_{\alpha\beta}^{\text{Ir.}(2)} = \gamma_{\alpha\beta}^{\text{Ir.}(1)}$. Therefore, if $\Lambda_{\beta}^{(2)\alpha}, \Lambda_{\beta}^{(1)\alpha}$ are the simplifying automorphisms, i.e. $\gamma_{\mu\nu}^{(2)} = \Lambda_{\mu}^{(2)\alpha}\Lambda_{\nu}^{(2)\beta}\gamma_{\alpha\beta}^{\text{Ir.}(2)}$ and $\gamma_{\mu\nu}^{(1)} = \Lambda_{\mu}^{(1)\alpha}\Lambda_{\nu}^{(1)\beta}\gamma_{\alpha\beta}^{\text{Ir.}(1)}$, then the transformation $\Lambda_{\beta}^{\alpha} = (\Lambda^{-1})_{\rho}^{(1)\alpha}\Lambda_{\beta}^{(2)\rho}$ connects $\gamma_{\mu\nu}^{(2)}$ to $\gamma_{\mu\nu}^{(1)}$ and obviously belongs to $Aut(G)$. □

Returning to the form of the solutions to equations (2.23), it is straightforward to check that every scalar combination of $C_{\mu\nu}^{\lambda}$ and $\gamma_{\mu\nu}, \gamma^{\alpha\beta}$ is annihilated by all $X_{(i)}$. The number of such independent scalar contractions is at most 6: the 10 $\gamma_{\mu\nu}$ plus the 12 $C_{\mu\nu}^{\lambda}$ (24 initially independent minus 12 independent Jacobi identities) minus 16 arbitrary elements of $GL(4, \mathbb{R})$.

The same number is obtained by observing that the automorphism group always contains the inner automorphism subgroup which has four generators, thus there will be at most $10-4=6$ independent scalar combinations.

A common, though not unique, suitable basis in the space of all such scalars, valid for all four-dimensional real Lie algebras, is

$$q^1 \quad q^2 \quad q^3 \quad q^4 \quad q^5 \quad q^6 \tag{2.25a}$$

where

$$q^1 = \Pi_{\alpha\beta\mu\nu}\gamma^{\alpha\mu}\gamma^{\beta\nu} \tag{2.25b}$$

$$q^2 = C^{\alpha}_{\beta\kappa}C^{\beta}_{\alpha\lambda}\gamma^{\kappa\lambda} \tag{2.25c}$$

$$q^3 = \Pi_{\alpha\beta\mu\nu}\Pi^{\alpha\beta\mu\nu} \tag{2.25d}$$

$$q^4 = \Pi_{\alpha\beta\kappa\lambda}\Pi_{\mu\nu\rho\sigma}\Pi^{\alpha\beta\mu\rho}\gamma^{\kappa\nu}\gamma^{\lambda\sigma} \tag{2.25e}$$

$$q^5 = \Upsilon_\alpha\Upsilon_\beta\gamma^{\alpha\beta} \tag{2.25f}$$

$$q^6 = \Omega_\alpha\Omega_\beta\gamma^{\alpha\beta} \tag{2.25g}$$

with the allocations:

$$\Pi_{\alpha\beta\mu\nu} = C^{\rho}_{\alpha\beta}C^{\sigma}_{\mu\nu}\gamma_{\rho\sigma} \tag{2.26}$$

$$\Upsilon_\alpha = \Pi_{\alpha\beta\mu\nu}C^{\nu}_{\kappa\lambda}\gamma^{\beta\lambda}\gamma^{\mu\kappa} \tag{2.27}$$

$$\Omega_\alpha = \Pi_{\alpha\beta\mu\nu}C^{\nu}_{\kappa\lambda}\Pi^{\beta\lambda\mu\kappa}. \tag{2.28}$$

(Greek indices are raised and lowered with $\gamma^{\alpha\beta}$ and $\gamma_{\alpha\beta}$ respectively.)

The number of functionally independent q^As is 6 only for 5 out of the 30 homogeneous 4-spaces considered here—a fact that is reminiscent of the analogous situation in Bianchi types, where only types VIII and IX possess three functionally independent qs. For the rest of the cases, the number of functionally independent q^A is less than 6.

These q^A can serve as 'coordinates' of the reduced configuration space on which the Wheeler–DeWitt equation is to be founded.

3. Discussion

We have investigated the action of the automorphism group of all four-dimensional, real Lie algebras in the space of 4×4, real, symmetric and positive definite matrices, which is the configuration space of homogeneous 4-spaces (admitting an invariant basis of 1-forms). These automorphisms naturally emerge as the non-trivial, well defined action of the diffeomorphism group on this space. The finite invariant description of these 4-spaces is given in terms of the scalar combinations of the structure constants with the scale factor matrix. The number of such independent scalars is found to be at most 6. These scalars can be considered either as the independent solutions to (2.23) or as the differential scalar contractions constructed out of the metric and its derivatives.

In three dimensions these are the curvature and/or higher derivative curvature scalars; in four dimensions there are also scalar combinations of the metric and its derivatives of degree greater than 3 (metric invariants [11]), which cannot be expressed as higher derivative curvature invariants.

The q^As given in (2.25*b*) may be curvature, higher derivative curvature or metric invariants. They irreducibly characterize the corresponding homogeneous space. These quantities are useful both at the classical and quantum level: classically the irreducibles of the scale factor matrix (which in the light of the theorem are essentially the q^As) can be used as a starting point for solving the Einstein field equations of the corresponding five-dimensional cosmological models. Quantum mechanically the Wheeler–DeWitt equation, when an action principle exists, is to be constructed on the reduced configuration space spanned by the q^As. The reasoning behind this is the desire to have 'gauge' invariant wavefunctions.

Our analysis does not treat the Kantowski–Sacks-like geometries (homogeneous spaces with multiple transitive groups of motions). These are 5 in number [12].

Finally, we would like to mention a word about the computation of the basis of invariants: it can be carried out using a symbolic algebra package (such as Mathematica) and the hard thing is to find a basis valid for all 30 homogeneous spaces. The price one pays is that one has to consider as many as ten powers of the structure constants (i.e. ten derivatives of the metric . This, however, is a worthwhile sacrifice of simplicity, since it will enable one to make comparative studies of the corresponding five-dimensional quantum cosmologies.

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Appendix

We give the automorphisms and their generators for $A_{3,8} \oplus A_1$ and $A_{3,9} \oplus A_1$ Lie algebras, in symbolic (matrix) notation.

$A_{3,8} \oplus A_1$:

$$\Lambda = \text{Rotation}_{xy}\text{Boost}_{xz}\text{Boost}_{yz}C \tag{A.1}$$

where

$$\text{Rotation}_{xy} = \begin{pmatrix} \cos(a_1) & \sin(a_1) & 0 & 0 \\ -\sin(a_1) & \cos(a_1) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{A.2}$$

$$\text{Boost}_{xz} = \begin{pmatrix} \cosh(a_2) & 0 & \sinh(a_2) & 0 \\ 0 & 1 & 0 & 0 \\ \sinh(a_2) & 0 & \cosh(a_2) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{A.3}$$

$$\text{Boost}_{yz} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cosh(a_3) & \sinh(a_3) & 0 \\ 0 & \sinh(a_3) & \cosh(a_3) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{A.4}$$

$$C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & a_4 \end{pmatrix}. \tag{A.5}$$

The generator:

$$\begin{pmatrix} 0 & g_2 & g_3 & 0 \\ -g_2 & 0 & g_7 & 0 \\ g_3 & g_7 & 0 & 0 \\ 0 & 0 & 0 & g_{16} \end{pmatrix}. \tag{A.6}$$

$A_{3,9} \oplus A_1$:

$$\Lambda = \text{Rotation}_{xy}\text{Rotation}_{xz}\text{Rotation}_{yz}C \tag{A.7}$$

where

$$\text{Rotation}_{xy} = \begin{pmatrix} \cos(a_1) & \sin(a_1) & 0 & 0 \\ -\sin(a_1) & \cos(a_1) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{A.8}$$

$$\text{Rotation}_{xz} = \begin{pmatrix} \cos(a_2) & 0 & -\sin(a_2) & 0 \\ 0 & 1 & 0 & 0 \\ \sin(a_2) & 0 & \cos(a_2) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{A.9}$$

$$\text{Rotation}_{yz} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(a_3) & \sin(a_3) & 0 \\ 0 & -\sin(a_3) & \cos(a_3) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{A.10}$$

$$C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & a_4 \end{pmatrix}. \tag{A.11}$$

The generator:

$$\begin{pmatrix} 0 & g_2 & g_3 & 0 \\ -g_2 & 0 & g_7 & 0 \\ -g_3 & -g_7 & 0 & 0 \\ 0 & 0 & 0 & g_{16} \end{pmatrix}. \tag{A.12}$$

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